Claremont Center for the Mathematical Sciences

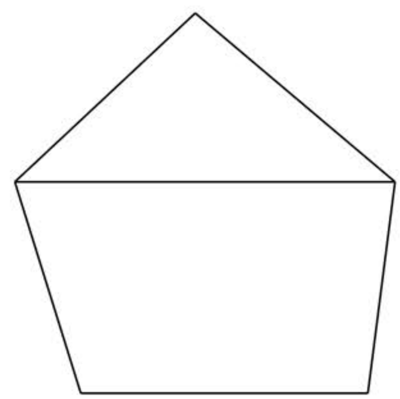
CCMS Math Challenge - H Problem Set 3 - December 1, 2015

High school students are invited to submit solutions by the end of the month in which the challenge is first published. The problems are **difficult**; an easier set is also available.

Details at: **ccms.claremont.edu/mc**

1. A club with seven members wants to form a number of three-person committees, but they require that no two committees should have more than one member in common. For example, if the members of the club are A, B, C, D, E, F and G, they could form these five committees: {A,B,G}, {A,C,F}, {A,D,E}, {B,C,E} and {D,F,G}. What is the maximum number of committees the club can form? Prove that your answer is correct.

2. Let us say that a diagonal of a pentagon is *good* if it is parallel to one of the sides of the pentagon. Show that if four of the five diagonals of a pentagon are good, then the fifth diagonal is also good.



3. Jake tells Jenny that he has three children, two of whom are twins, and that the ages of all three children are integers. Jake also tells Jenny the sum and the product of the ages of his children. Jenny then says that the age of the non-twin child must be either 9 or 25, but that there is not enough information to determine which of these is correct. Determine (with proof) the product of the ages of Jake's children. [This is a corrected version of the original problem.]

4. Let $S = \{2, 3, 22, 23, 32, 33, 222, \ldots\}$ be the set of all positive integers whose decimal digits are 2s and 3s only. Show that no three distinct members of S are in arithmetic progression. (Recall that three integers $a \leq b \leq c$ are said to be in *arithmetic progression* if $c - b = b - a$.)

5. Show that for each odd prime number p, there is exactly one positive integer n such that $n(n+p)$ is a perfect square.

6. Suppose that for each integer $k \geq 1$, we have an unlimited supply of rectangular $2 \times k$ tiles. Given an integer $n \geq 1$, write $a(n)$ to denote the number of ways that a $2 \times n$ rectangle can be covered using our tiles. It is clear, for example, that $a(1) = 1$, and a little experimentation shows that $a(2) = 3$ and $a(3) = 6$. Compute $a(7)$.